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# Multiple Acceptable Solutions in Structural Model Improvement

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**This paper examines the question as to whether it is possible to obtain the correct discrete linear analytical model of an actual structure by the use of analysis and test data. It is shown that no such correct model exists. An illustration of an application is shown where multiple acceptable and very different analytical models are all representative of the structure. The implications regarding methods of improving analytical models and detecting damage are discussed.**

## Introduction

EVERY engineer is aware of the importance of the current capabilities of finite element analysis in the design of complex structures. The ability to correctly model a particular structure is vital to the integrity of the product. The limiting factor on these capabilities is the correctness of the analytical model. In spite of the great advances in problem size and element sophistication, an analytical model cannot be unconditionally accepted without some level of validation making use of data from tests on the actual structure.

This consideration has led to numerous efforts in recent years to do more than simply validate an analytical model, that is, to use test data to actually correct or improve this model. The object of this approach is to combine the best of both worlds: the theoretical, intuitive understanding of the phenomena; and real life data from the actual structure without any of the limiting assumptions made in the analysis.

Unfortunately, this concept, which seems quite straightforward, has a number of problems that are often overlooked. Some of these are discussed in some detail in Refs. 1 and 2. This paper will concentrate on the question of identifying a unique linear finite element model (FEM) of a real structure. The theoretical reasons why this is not possible and the practical implications will be discussed. An example will be shown that illustrates the fact that it is possible to have a large (actually, infinite) number of reasonable and varied improved models, all of which are consistent with both the analysis and the test data.

In the following discussion, the FEM and a valid representation of the actual structure will be assumed to be of the general form

$$(K - \omega^2 M)y = f \tag{1}$$

consisting of a mass matrix $M$ and a stiffness matrix $K$. The displacements of the degrees of freedom $y$ are the steady-state responses produced by the forcing vector $f$ acting at a single frequency $\omega$. It is believed that the simplifying assumptions that the structure is undamped and linear and has a finite, but larger number of degrees of freedom (DOF) than the model will not affect the conclusions to be drawn from this study.

## Linear Model of a Linear Structure

Any structure has an infinite number of degrees of freedom. A structure with linear characteristics has an infinite number of normal modes and natural frequencies. A linear analytical model with a finite number of DOF cannot be a true representation of a structure, since it has only a finite number of modes. Although this statement is obviously true, it is worthwhile to look at this analytically and illustrate the effect to show that this is not a trivial consideration.

Consider the equations of a structure and an FEM of the same structure with fewer DOF. Equation (1) can be written for the structure in partitioned form, where the subscript 1 refers to the reduced set of DOF of the analytical model

$$\left\{\begin{vmatrix} K_1 & K_2 \\ K_2^T & K_4 \end{vmatrix} - \omega^2 \begin{vmatrix} M_1 & M_2 \\ M_2^T & M_4 \end{vmatrix}\right\} \begin{vmatrix} y_1 \\ y_2 \end{vmatrix} = \begin{vmatrix} f_1 \\ f_2 \end{vmatrix} \tag{2}$$

From this relationship, it is possible to write the corresponding equation of the model, where it is assumed that forces are applied only at the model degrees of freedom (i.e., $f_2 = 0$)

$$Z_1 y_1 = f_1 \tag{3}$$

where

$$Z_1 = \left(K_1 - \omega^2 M_1\right) - \left(K_2 - \omega^2 M_2\right) \times \left(K_4 - \omega^2 M_4\right)^{-1}\left(K_2^T - \omega^2 M_2^T\right) \tag{4}$$

$Z_1$ is a valid model of the structure. Note, however, that $Z_1$ cannot be reduced to the form of Eq. (1). Various formulations in terms of equivalent mass and stiffness matrices are possible but either or both must be functions of frequency. A common approach is to treat the stiffness as a static quantity and use the reduction of Turner et al.[3] and then use the well-known Guyan[4] reduction for the mass. In this scenario the reduced stiffness is considered to be exact, but the mass is valid only at $\omega = 0$. These issues are discussed in more detail in Refs. 2 and 5.

Whereas it is obvious, from the preceding, that a linear model with a finite number of degrees of freedom cannot completely represent a linear continuous structure, it is probably reasonable to assume that such a model may be a good representation over a limited frequency range. There is another important consideration, however, that emphasizes the limitations on this problem.

Consider that one measures normal modes at a limited number of points on the structure or computes them using the derived or improved analytical model. If one compares them with the continuous modes of the structure, it is to be expected that the shape of the low-order modes will be reasonably well represented by the limited measurements. However, as the order of the mode approaches the number of DOF or the number of measurements, it is quite apparent that the discretized mode will not be representative of the mode shape, and such features as sign changes and peaks could and would be missing. One cannot expect that the measured modes would have the required intermodal characteristics, as, for example, orthogonality with respect to the mass matrix.

If one were to use a procedure to adjust the mass and stiffness matrices to make the $n$ measured modes of an $n$-DOF model orthogonal, it is quite apparent that these matrices would be extremely unrealistic.

Thus, it may be concluded that a linear $n$th-order discrete model of a continuous structure will have a limited valid range of frequencies that is significantly below the $n$th natural frequency of the structure or the model. It is also apparent that modes measured at $n$ DOF that approach the $n$th mode of the structure should not be used in a procedure to improve an analytical model since they poorly

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represent the characteristics of the modes. In addition, if an FEM of order $n$ predicted the modes of the structure up to or close to mode $n$, it would be a poor representation of the physical characteristics of the structure.

## Multiple Models

Assuming that the mass matrix is known, it is possible to express the stiffness matrix of the analytical model in terms of the modes of the system, as follows:

$$K = M\Phi\left[\omega_i^2/m_i\right]\Phi^T M \tag{5}$$

where $\Phi$ is the matrix of normal modes, $\omega_i$ and $m_i$ are the frequency and generalized mass of the $i$th mode (on the diagonal matrix). $K$ can be of full rank only when all $n$ independent modes are included. As discussed earlier, only the lower frequency modes should agree with the measured modes. Since there is no restriction on the higher modes, any set can be used, as long as all of the modes are orthogonal, and the high frequencies are such that they have no significant affect on the low-frequency response.

In techniques to obtain the correct model, it is a common approach to include data based on the analytical formulation in addition to the test data. This may be considered to supply the information necessary to complete the unique definition of the problem. In actuality, however, this information does not contribute to the uniqueness of the solution.

To illustrate this effect, consider two general and common approaches. These are 1) select a limited set of parameters and find the changes in the values necessary to make the model match the test data (i.e., see Ref. 6) and 2) find the smallest necessary change in the stiffness and or mass matrices (i.e., see Refs. 7 and 8).

In the case of approach 1, choosing a different set of parameters to vary will yield different corrected models. In case 2, choosing a different criteria for defining smallest changes will result in a different model of the structure. Also, there is no logical basis for assuming that the smallest changes, by whatever criteria, are the correct ones. In both cases, if properly carried out, all solutions would precisely predict the test results within the limited frequency range of the lower modes.

Thus, it must be concluded that there is no true or unique solution to the problem of finding the correct linear model of the structure.

## Observation Regarding the Multiple Models

From Eq. (5) it is apparent that the dominant terms in the stiffness matrix are due to the highest frequency modes of the model (since $\omega_i^2$ is in the numerator). It is important to note, as already discussed, that these high-frequency modes are not the same as the modes of the actual structure. Since it is only the lower modes that are meaningful, there may be large numerical variations in acceptable stiffness matrices for modeling a structure.

The inverse of the stiffness matrix, the static influence coefficient matrix, can be written

$$C = K^{-1} = \Phi\left[1/\left(\omega_i^2 m_i\right)\right]\Phi^T \tag{6}$$

Note that this matrix, which describes the static response per unit force, is dominated by the lower frequency modes of the model. All acceptable models of a structure, must have lower modes of nearly the same shapes and frequencies as the structure, and one should expect little variation in the influence coefficient matrices.

Thus, one may make the seemingly paradoxical observation that whereas the acceptable stiffness matrices may vary significantly, all their inverses must be nearly identical (see Ref. 9. for further discussion).

## Illustrative Example

To demonstrate these conclusions, consider the following simple example of modeling a structure. The analytical model of the structure is represented by a system consisting of six masses, each having a single degree of freedom. A schematic representation of this model is shown in Fig. 1. Note that the load paths, represented by weightless linear springs, connect all of the degrees of freedom, as in any real structure. Each of the masses is also connected to the ground by a spring.

It is assumed that all of the mass elements have a known value of unity. The spring rates are unknown, but engineering estimates should be available. It is also assumed that the first four normal modes and natural frequencies have been determined through testing on the structure. These data are shown in Table 1.

Using Eq. (5) it is possible to form an infinite number of meaningful stiffness matrices (i.e., having physically meaningful spring rates), all of which will exactly predict the test results which are shown in Table 1. The stiffness matrices were formed using the first four known modes and frequencies and two additional orthogonal modes. The frequencies of these two modes were varied so as to yield physically meaningful results. A sampling of these results is shown in Table 2.

**Table 1 Simulated measured modes**

| Mode no. | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Natural freq, Hz | 0.471 | 1.511 | 2.013 | 2.421 |
| Element no. | Modal displacements | | | |
| 1 | 0.820 | −0.547 | 1.000 | −0.712 |
| 2 | 0.910 | −0.430 | 0.065 | 1.000 |
| 3 | 0.936 | −0.272 | −0.235 | 0.312 |
| 4 | 0.960 | −0.078 | −0.580 | −0.918 |
| 5 | 0.977 | 0.175 | −0.707 | −0.688 |
| 6 | 1.000 | 1.000 | 0.588 | 0.225 |

**Table 2 Valid spring rates**

| | Connecting spring rates[b] | | | | | | |
|---|---|---|---|---|---|---|---|
| Elements[a] | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 1,1 | 25 | 25 | 25 | 25 | 25 | 25 | 25 |
| 1,2 | 50 | 50 | 51 | 51 | 51 | 51 | 51 |
| 1,3 | 50 | 49 | 48 | 47 | 47 | 47 | 47 |
| 1,4 | 10 | 11 | 12 | 12 | 12 | 12 | 12 |
| 1,5 | 5 | 5 | 4 | 4 | 5 | 5 | 5 |
| 1,6 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| 2,2 | 10 | 10 | 10 | 10 | 10 | 10 | 10 |
| 2,3 | 150 | 144 | 136 | 132 | 123 | 116 | 112 |
| 2,4 | 10 | 15 | 22 | 26 | 16 | 8 | 4 |
| 2,5 | 10 | 8 | 6 | 5 | 16 | 25 | 30 |
| 2,6 | 5 | 5 | 5 | 5 | 4 | 3 | 3 |
| 3,3 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| 3,4 | 200 | 185 | 163 | 152 | 162 | 171 | 175 |
| 3,5 | 20 | 25 | 32 | 35 | 24 | 14 | 10 |
| 3,6 | 10 | 10 | 9 | 9 | 10 | 11 | 12 |
| 4,4 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| 4,5 | 150 | 146 | 140 | 137 | 124 | 114 | 109 |
| 4,6 | 15 | 15 | 16 | 16 | 17 | 18 | 19 |
| 5,5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| 5,6 | 50 | 50 | 50 | 50 | 48 | 47 | 46 |
| 6,6 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |

[a]Spring rates between elements, $n, n$ is between $n$ and ground.
[b]Each column represents a valid model, numbers are rounded for ease of viewing.

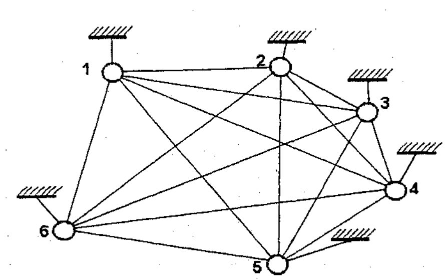


**Fig. 1 Schematic of model of structure showing load paths connecting elements, each of which has a single DOF.**

**Table 3 Illustrative stiffness matrices**

| Stiffness matrix $K$ corresponding to column 1, Table 2 | | | | | |
|---|---|---|---|---|---|
| 145.0 | −50.02 | −50.00 | −9.990 | −4.976 | −5.016 |
| | 235.0 | −150.0 | −10.01 | −10.01 | −5.002 |
| | | 435.0 | −20.00 | −20.01 | −9.997 |
| sym | | | 390.0 | −150.0 | −14.99 |
| | | | | 240.0 | −50.00 |
| | | | | | 90.02 |
| **Stiffness matrix $K$ corresponding to column 7, Table 2** | | | | | |
| 144.9 | −51.47 | −46.74 | −11.78 | −4.908 | −4.982 |
| | 209.5 | −112.0 | −3.921 | −29.92 | −2.516 |
| | | 360.6 | −174.7 | −9.627 | −11.99 |
| sym | | | 322.9 | −10.85 | −18.85 |
| | | | | 203.9 | −46.24 |
| | | | | | 89.61 |

**Table 4 Valid spring rates**

| | Connecting spring rates[b] | | | | | | |
|---|---|---|---|---|---|---|---|
| Elements[a] | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 1, 1 | 25 | 25 | 25 | 25 | 25 | 25 | 25 |
| 1, 2 | 54 | 54 | 54 | 54 | 53 | 53 | 53 |
| 1, 3 | 39 | 40 | 40 | 41 | 41 | 42 | 43 |
| 1, 4 | 18 | 17 | 17 | 17 | 16 | 16 | 15 |
| 1, 5 | 3 | 3 | 3 | 4 | 4 | 4 | 4 |
| 1, 6 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| 2, 2 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |
| 2, 3 | 58 | 61 | 64 | 67 | 70 | 73 | 80 |
| 2, 4 | 38 | 36 | 33 | 31 | 28 | 25 | 19 |
| 2, 5 | 25 | 26 | 27 | 28 | 29 | 29 | 31 |
| 2, 6 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3, 3 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| 3, 4 | 56 | 64 | 72 | 80 | 88 | 97 | 114 |
| 3, 5 | 41 | 38 | 36 | 33 | 31 | 28 | 23 |
| 3, 6 | 11 | 11 | 11 | 11 | 11 | 11 | 12 |
| 4, 4 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| 4, 5 | 64 | 66 | 68 | 70 | 73 | 75 | 80 |
| 4, 6 | 22 | 22 | 22 | 21 | 21 | 21 | 21 |
| 5, 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| 5, 6 | 44 | 44 | 44 | 45 | 45 | 45 | 45 |
| 6, 6 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |

[a]Spring rates between elements, $n$, $n$ is between $n$ and ground.
[b]Each column represents a valid model, numbers are rounded for ease of viewing.

In formulating Table 2, it was assumed that the ranges of the spring rates should fall within a particular physically reasonable range. Between each column in the table lie a continuous range of values, all of which will exactly yield the first four measured modes and natural frequencies, as shown in Table 1. Note that the values shown in the table have been rounded for ease of viewing.

The significant differences in the stiffness matrices corresponding to the first and last column of Table 2 are illustrated in Table 3. Note that in this simple model the spring stiffness connecting DOFs $i$ and $j$ appears as the negative of element $i, j$ in the stiffness matrix. The elements in Table 3 are shown to more significant figures than in Table 2.

To illustrate the fact that a broad range of meaningful models exist which behave like the structure, additional data is presented in Tables 4 and 5 where different limits on the spring rates were imposed. All of the models represented by the data in Tables 2–5 will precisely satisfy the requirements for structural realism and the matching of the test results.

It has already been stated, that although the stiffness matrices may vary significantly in magnitude, their inverses should vary only slightly. The four matrices in Tables 3 and 5 were inverted. The results were very nearly identical, and the averaged values are shown in Table 6. The average variation in each element was approximately 0.4% and the maximum variation in any one element was 2.7%.

**Table 5 Illustrative stiffness matrices**

| Stiffness matrix $K$ corresponding to column 1, Table 4 | | | | | |
|---|---|---|---|---|---|
| 144.5 | −54.02 | −39.43 | −17.87 | −3.153 | −5.066 |
| | 187.1 | −57.92 | −38.25 | −25.20 | −2.306 |
| | | 211.8 | −56.91 | −40.58 | −10.76 |
| sym | | | 203.5 | −63.93 | −21.84 |
| | | | | 182.0 | −44.43 |
| | | | | | 89.45 |
| **Stiffness matrix $K$ corresponding to column 7, Table 4** | | | | | |
| 144.7 | −52.94 | −42.68 | −15.00 | −4.061 | −5.015 |
| | 194.3 | −79.50 | −19.17 | −31.23 | −1.973 |
| | | 276.5 | −114.1 | −22.50 | −11.75 |
| sym | | | 254.1 | −79.92 | −20.96 |
| | | | | 187.1 | −44.71 |
| | | | | | 89.46 |

**Table 6 Inverse of stiffness matrices[a]**

| | | | | | |
|---|---|---|---|---|---|
| 2.06 | 1.67 | 1.66 | 1.62 | 1.58 | 1.54 |
| | 2.21 | 1.88 | 1.83 | 1.81 | 1.73 |
| | | 2.20 | 1.97 | 1.87 | 1.83 |
| sym | | | 2.31 | 2.05 | 1.94 |
| | | | | 2.45 | 2.09 |
| | | | | | 2.98 |

[a]Averaged values of the inverses of matrices in Tables 3 and 5, average variation = 0.4%, max variation = 2.7%, all values $\times 10^{-2}$.

## Conclusions

It has been shown, from a theoretical viewpoint and through an illustrative example, that there is no such thing as the correct linear finite element model of a linear structure. There are, however, many physically reasonable models which will behave like the structure over a limited regime.

For those engineers who are seeking to simply validate the analytically derived model, they may gain confidence in the analysis by determining that there is a good model which deviates only slightly from the analysis.

For those who are seeking to modify the model to agree with the test data, the characteristics of the selected modified model must be investigated to determine if this model will properly behave like the structure for whatever purpose the engineer has in mind. For example, the model may be used to study the effects of different load distributions or the effects of structural changes, or to help design control system algorithms. In each case it must be demonstrated that the selected improvement will properly model these effects. Research is required in this area to allow the engineer to establish criteria for evaluating this characteristic of the selected improved model.

For those who are using such methods to identify damage by observing the changes in the structural characteristics, great care must be taken. For this application, one must be able to identify the approximate location of the true structural changes. Although much progress has been made in this area of research, one must be aware of the fact that just because a particular changed model is consistent with the test data, that in itself may not be enough to identify the damaged area.

The idea of using test data to improve or modify or update an analytical model is a concept with many potential benefits. To achieve these benefits it is important that the investigators study the problem and carry out the necessary research with full realization of the inherent limitations.

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